

Einstein frame or Jordan frame ?

Valerio Faraoni¹ and Edgard Gunzig^{1,2}

¹ *Research Group in General Relativity*

Université Libre de Bruxelles, Campus Plaine CP 231

Blvd. du Triomphe, 1050 Bruxelles, Belgium

² *Instituts Internationaux de Chimie et de Physique Solvay*

Abstract

Scalar–tensor theories of gravity can be formulated in the Jordan or in the Einstein frame, which are conformally related. The issue of which conformal frame is physical is a contentious one; we provide a straightforward example based on gravitational waves in order to clarify the issue.

Published in *Int. J. Theor. Phys.* **38**, 217.

1 Introduction

Scalar–tensor theories of gravity, of which Brans–Dicke [1] theory is the prototype, are competitors to Einstein’s theory of general relativity for the description of classical gravity. Renewed interest in scalar–tensor theories of gravity is motivated by the extended [2] and hyperextended [3] inflationary scenarios of the early universe. Additional motivation arises from the presence of a fundamental Brans–Dicke–like field in most high energy physics theories unifying gravity with the other interactions (the string dilaton, the supergravity partner of spin 1/2 particles, etc.).

It is well known since the original Brans–Dicke paper [1] that two formulations of scalar–tensor theories are possible; the version in the so–called Jordan conformal frame commonly presented in the textbooks (e.g. [4]–[6]), and the less known version based on the Einstein conformal frame, which is related to the former one by a conformal transformation and a redefinition of the gravitational scalar field present in the theory. The possibility of two formulations related by a conformal transformation exists also for Kaluza–Klein theories and higher derivative theories of gravity (see [7, 13] for reviews). The problem of whether the two formulations of a scalar–tensor theory in the two conformal frames are equivalent or not has been the issue of lively debates, which are not yet settled, and often is the source of confusion in the technical literature. While many authors support the point of view that the two conformal frames are equivalent, or even that physics at the energy scale of classical gravity and classical matter is always conformally invariant, other authors support the opposite point of view, and others again are not aware of the problem (see the “classification of authors” in [7]). The issue is important in principle and in practice, since there are many applications of scalar–tensor theories and of conformal transformation techniques to the physics of the early universe and to astrophysics. The theoretical predictions to be compared with the observations (in cosmology, the existence of inflationary solutions of the field equations, and the spectral index of density perturbations) crucially depend on the conformal frame adopted to perform the calculations.

In addition, if the two formulations of a scalar–tensor theory are not equivalent, the problem arises of whether one of the two is physically preferred, and which one has to be compared with experiments and astronomical observations. Are both conformal versions of the same theory viable, and good candidates for the description of classical gravity ? Unfortunately many authors neglect these problems, and the issue is not discussed in the textbooks explaining scalar–tensor theories. On the other hand, it emerges from the work of several authors, in different contexts (starting with Refs. [8, 10, 11] on Kaluza–Klein and Brans–Dicke theories, and summarized in [7, 13]), that

1. the formulations of a scalar–tensor theory in the two conformal frames are physically inequivalent.
2. The Jordan frame formulation of a scalar–tensor theory is not viable because the energy density of the gravitational scalar field present in the theory is not bounded from below (violation of the weak energy condition [14]). The system therefore is unstable and decays toward a lower and lower energy state *ad infinitum* [7, 13]).
3. The Einstein frame formulation of scalar–tensor theories is free from the problem 2). However, in the Einstein frame there is a violation of the equivalence principle due to the anomalous coupling of the scalar field to ordinary matter (this violation is small and compatible with the available tests of the equivalence principle [11]; it is indeed regarded as an important low energy manifestation of compactified theories [9]–[11], [12]).

It is clear that property 2) is not acceptable for a viable *classical* theory of gravity (a quantum system, on the contrary, may have states with negative energy density [15, 16]). A classical theory must have a ground state that is stable against small perturbations.

In spite of this compelling argument, there is a tendency to ignore the problem, which results in a uninterrupted flow of papers performing computations in the Jordan frame. The use of the latter is also implicitly supported by most textbooks on gravitational theories. Perhaps this is due to reluctance in accepting a violation of the equivalence principle, on philosophical and aesthetic grounds, or perhaps it is due to the fact that the best discussions of this subject are rather mathematical than physical in character, and not well known. In this paper, we present a straightforward argument in favor of the Einstein frame, in the hope to help settling the issue.

In Sec. 2 we recall the relevant formulas. In Sec. 3 we present a simple argument based on scalar–tensor gravitational waves, and section 4 contains a discussion and the conclusions.

2 Conformal frames

The textbook formulation of scalar–tensor theories of gravity is the one in the Jordan conformal frame, in which the action takes the form¹

$$S = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left[f(\phi) R - \frac{\omega(\phi)}{\phi} g^{\alpha\beta} \nabla_\alpha \phi \nabla_\beta \phi + \Lambda(\phi) \right] + \int d^4x \sqrt{-g} \mathcal{L}_{matter} , \quad (2.1)$$

where \mathcal{L}_{matter} is the Lagrangian density of ordinary matter, and the couplings $f(\phi)$, $\omega(\phi)$ are regular functions of the scalar field ϕ . Although our discussion applies to the generalized theories described by the action (2.1), for simplicity, we will restrict ourselves to Brans–Dicke theory, in which ω and Λ are constants and we will omit the non–gravitational part of the action, which is irrelevant for our purposes. We further assume that $\Lambda = 0$; the field equations then reduce to

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{\omega}{\phi^2} \left(\nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} \nabla^\alpha \phi \nabla_\alpha \phi \right) + \frac{1}{\phi} (\nabla_\mu \nabla_\nu \phi - g_{\mu\nu} \square \phi) , \quad (2.2)$$

$$\square \phi + \frac{\phi R}{2\omega} = 0 . \quad (2.3)$$

It is well known since the original Brans–Dicke paper [1] that another formulation of the theory is possible: the conformal transformation

$$g_{\mu\nu} \longrightarrow \tilde{g}_{\mu\nu} = \phi g_{\mu\nu} , \quad (2.4)$$

and the scalar field redefinition

$$\phi \longrightarrow \tilde{\phi} = \int \frac{(2\omega + 3)^{1/2}}{\phi} d\phi , \quad (2.5)$$

(where $\omega > -3/2$), recast the theory in the so–called Einstein conformal frame², in which the gravitational part of the action becomes that of Einstein gravity plus a non

¹The metric signature is $-+++$, the Riemann tensor is given in terms of the Christoffel symbols by $R_{\mu\nu\rho}{}^\sigma = \Gamma_{\mu\rho,\nu}^\sigma - \Gamma_{\nu\rho,\mu}^\sigma + \Gamma_{\mu\rho}^\alpha \Gamma_{\alpha\nu}^\sigma - \Gamma_{\nu\rho}^\alpha \Gamma_{\alpha\mu}^\sigma$, the Ricci tensor is $R_{\mu\rho} \equiv R_{\mu\nu\rho}{}^\nu$, and $R = g^{\alpha\beta} R_{\alpha\beta}$. ∇_μ is the covariant derivative operator, $\square \equiv g^{\mu\nu} \nabla_\mu \nabla_\nu$, $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$, and we use units in which the speed of light and Newton’s constant assume the value unity.

²Also called “Pauli frame” in Refs. [9]–[11])

self-interacting scalar field³,

$$S = \int d^4x \sqrt{-\tilde{g}} \left[\frac{\tilde{R}}{16\pi} - \frac{1}{2} \tilde{g}^{\mu\nu} \tilde{\nabla}_\mu \tilde{\phi} \tilde{\nabla}_\nu \tilde{\phi} \right]. \quad (2.6)$$

The field equations are the usual Einstein equations with the scalar field as a source,

$$\tilde{R}_{\mu\nu} - \frac{1}{2} \tilde{g}_{\mu\nu} \tilde{R} = 8\pi \left(\tilde{\nabla}_\mu \tilde{\phi} \tilde{\nabla}_\nu \tilde{\phi} - \frac{1}{2} \tilde{g}_{\mu\nu} \tilde{\nabla}^\alpha \tilde{\phi} \tilde{\nabla}_\alpha \tilde{\phi} \right), \quad (2.7)$$

$$\Box \tilde{\phi} = 0. \quad (2.8)$$

It has been pointed out (see [7, 13] for reviews) that the Jordan frame formulation of Brans–Dicke theory is not viable because the sign of the kinetic term for the scalar field is not positive definite, and hence the theory does not have a stable ground state. The system decays toward lower and lower energy states without a lower bound. On the contrary, the Einstein frame version of the theory possesses the desired stability property. These features were first discovered in Kaluza–Klein and Brans–Dicke theory [8, 10, 11], and later ([7, 13] and references therein) in scalar–tensor and non-linear theories of gravity with Lagrangian density of the form $\mathcal{L} = f(\phi, R)$. Despite this difficulty with the energy, the textbooks still present the Jordan frame version of the theory without mention of its Einstein frame counterpart. The technical literature is also haunted by confusion on this topic, especially in cosmological applications [7, 13]. Many authors perform calculations in both conformal frames, while others support the use of the Jordan frame, or even claim that the two frames are physically equivalent. The issue of the conformal frame may appear a purely technical one, but it is indeed very important, in principle, and because the physical predictions of a classical theory of gravity, or of an inflationary cosmological scenario, are deeply affected by the choice of the conformal frame. Here, we study the violation of the weak energy condition by *classical* gravitational waves. It appears very hard to argue with the energy argument that leads to the choice of the Einstein frame [7]; moreover, the entire realm of classical⁴ physics is not conformally invariant. The literature on the topic is rather mathematical and abstract, and can be easily missed by the physically-minded reader. In the next section we propose a physical illustration of how the weak energy condition is violated in the Jordan frame, but not in the Einstein frame.

³If the Lagrangian density \mathcal{L}_{matter} of ordinary matter is included in the original action, it will appear multiplied by a factor $\exp(-\alpha\tilde{\phi})$ in the action (2.6); this anomalous coupling is responsible for a violation of the equivalence principle in the Einstein frame [9]–[11], [12]).

⁴Quantum states can violate the weak energy condition [15, 16]; in this paper, we restrict to classical gravitational theories.

3 Gravitational waves in the Jordan and in the Einstein frame

We begin by considering gravitational waves in the Jordan frame version of Brans–Dicke theory. In a locally freely falling frame, the metric and the scalar field are decomposed as follows

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} , \quad (3.1)$$

$$\phi = \phi_0 + \varphi , \quad (3.2)$$

where $\eta_{\mu\nu}$ is the Minkowski metric, ϕ_0 is constant, and the wave-like perturbations $h_{\mu\nu}$, φ/ϕ_0 have the same order of magnitude,

$$\mathcal{O}\left(\frac{\varphi}{\phi_0}\right) = \mathcal{O}(h_{\mu\nu}) = \mathcal{O}(\epsilon) \quad (3.3)$$

in terms of a smallness parameter ϵ . The linearized field equations in the Jordan frame

$$R_{\mu\nu} = \frac{\partial_\mu \partial_\nu \varphi}{\phi_0} , \quad (3.4)$$

$$\square \varphi = 0 , \quad (3.5)$$

allow the expansion of φ in monochromatic plane waves:

$$\varphi = \varphi_0 \cos(k_\alpha x^\alpha) , \quad (3.6)$$

where φ_0 is constant and $\eta_{\mu\nu} k^\mu k^\nu = 0$. Now note that, for any timelike vector ξ^μ , the quantity $T_{\mu\nu} \xi^\mu \xi^\nu$ (which represents the energy density of the waves as seen by an observer with four-velocity ξ^μ) is given, to the lowest order, by

$$T_{\mu\nu} \xi^\mu \xi^\nu = - (k_\mu \xi^\mu)^2 \frac{\varphi}{\phi_0} . \quad (3.7)$$

This quantity oscillates, changing sign with the frequency of φ and therefore violating the weak energy condition [14]. In addition, the energy density is not quadratic in the first derivatives of the field, and this implies that the energy density of the scalar field φ is of order $\mathcal{O}(\epsilon)$, while the contribution of the tensor modes $h_{\mu\nu}$ is only of order $\mathcal{O}(\epsilon^2)$ (and is given by the Isaacson effective stress–energy tensor [5] $T_{\mu\nu}^{(eff)}[h_{\alpha\beta}]$). The Jordan frame formulation of Brans–Dicke theory somehow discriminates between scalar and tensor modes. From an experimental point of view, this fact has important consequences for

the amplification induced by scalar–tensor gravitational waves on the light propagating through them, and for ongoing *VLBI* observations [17]–[19]. If the Jordan frame formulation of scalar–tensor theories was the physical one, astronomical observations could potentially detect the time–dependent amplification induced by gravitational waves in a light beam, which is of order ϵ [17]. If instead the Einstein frame formulation of scalar–tensor theories is physical (which is the case, as we shall see in the following), then the amplification effect is of order ϵ^2 , and therefore undetectable [17]–[19].

We now turn our attention to gravitational waves in the Einstein frame version of Brans–Dicke theory. The metric and scalar field decompositions

$$\tilde{g}_{\mu\nu} = \eta_{\mu\nu} + \tilde{h}_{\mu\nu} , \quad (3.8)$$

$$\tilde{\phi} = \tilde{\phi}_0 + \tilde{\varphi} , \quad (3.9)$$

where $\tilde{\phi}_0$ is constant and $O(\tilde{h}_{\mu\nu}) = O(\tilde{\varphi}/\tilde{\phi}_0) = O(\epsilon)$, lead to the equations

$$\tilde{R}_{\mu\nu} - \frac{1}{2} \tilde{g}_{\mu\nu} \tilde{R} = 8\pi \left(\tilde{T}_{\mu\nu}[\tilde{\varphi}] + \tilde{T}_{\mu\nu}^{(eff)}[\tilde{h}_{\mu\nu}] \right) , \quad (3.10)$$

$$\Box \tilde{\varphi} = 0 . \quad (3.11)$$

Here $\tilde{T}_{\mu\nu}[\tilde{\varphi}] = \partial_\mu \tilde{\varphi} \partial_\nu \tilde{\varphi} - \eta_{\mu\nu} \partial^\alpha \tilde{\varphi} \partial_\alpha \tilde{\varphi} / 2$. Again, we consider plane monochromatic waves

$$\tilde{\varphi} = \tilde{\varphi}_0 \cos(l_\alpha x^\alpha) , \quad (3.12)$$

where $\tilde{\varphi}_0$ is a constant and $\eta_{\mu\nu} l^\mu l^\nu = 0$. The energy density measured by an observer with timelike four–velocity ξ^μ in the Einstein frame is

$$\tilde{T}_{\mu\nu} \xi^\mu \xi^\nu = [l_\mu \xi^\mu \tilde{\varphi}_0 \sin(l_\alpha x^\alpha)]^2 + \tilde{T}_{\mu\nu}^{(eff)}[\tilde{h}_{\alpha\beta}] \xi^\mu \xi^\nu , \quad (3.13)$$

which is positive definite. The contributions of the scalar and tensor modes to the total energy density have the same order of magnitude $O(\epsilon^2)$, and are both quadratic in the first derivatives of the fields. The weak energy condition is satisfied in the Einstein, but not in the Jordan frame; physically reasonable matter in the classical domain is expected to satisfy the energy conditions [14].

Let us return for a moment to the Jordan frame: analogously to eq. (3.7), the energy–momentum 4–current density of scalar gravitational waves in the Jordan frame is

$$T_{0\mu} = -k_\mu (k_\nu \xi^\nu) \frac{\varphi}{\phi_0} . \quad (3.14)$$

In the Jordan frame, the energy density and current of spin 0 gravitational waves average to zero on time intervals much longer than the period of the waves. However, this is not a solution to the problem, since one can conceive of scalar gravitational waves with very long period. For example, gravitational waves from astronomical binary systems have periods ranging from hours to months (waves from μ -Sco, e.g., have period $3 \cdot 10^5$ s). The violation of the weak energy condition over such macroscopic time scales is unphysical.

4 Discussion and conclusions

The violation of the weak energy condition by scalar-tensor theories formulated in the Jordan conformal frame makes them unviable descriptions of classical gravity. Due to the fact that scalar dilatonic fields are ubiquitous in superstring and supergravity theories, there is a point in considering Brans-Dicke theory (and its scalar-tensor generalizations) as toy models for string theories (e.g. [9, 10, 20]), and in this case our considerations should be reanalyzed, because negative energy states are not forbidden at the quantum level [15, 16]. However, this context is quite limited, and differs from the usual classical studies of scalar-tensor theories.

The reluctance of the gravitational physics community in accepting the energy argument in favor of the Einstein frame is perhaps due to the fact that it was formulated in a rather abstract way. The example illustrated in the present paper shows, in a straightforward way, the violation of the weak energy condition by wave-like gravitational fields in Brans-Dicke theory formulated in the Jordan frame, and the viability of the Einstein frame counterpart of the same theory. The example is not academic, since a infrared catastrophe for scalar gravitational waves would have many observational consequences. One example studied in the astronomical literature consists of the amplification effect induced by scalar-tensor gravitational waves on a light beam, which differs in the Jordan and in the Einstein frame [17]–[19].

The argument discussed in this paper for Brans-Dicke theory can be easily generalized to other scalar-tensor theories. Our conclusions agree with, and are complementary to, those of Refs. [9]–[11], although our approach is different. It has also been pointed out that the Einstein frame variables $(\tilde{g}_{\mu\nu}, \tilde{\phi})$, but not the Jordan frame variables $(g_{\mu\nu}, \phi)$, are appropriate for the formulation of the Cauchy problem [21].

The example presented in this paper agrees with recent studies of the gravitational collapse to black holes in Brans-Dicke theory [22]. The noncanonical form of the stress-energy tensor of the Jordan frame Brans-Dicke scalar is responsible for the violation of the null energy condition ($R_{\alpha\beta}n^\alpha n^\beta \geq 0$ for all null vectors n^α). This causes a

decrease in time of the area of the black hole horizon [22], contrarily to the behaviour predicted by black hole thermodynamics [14] in general relativity. In our example, the null energy condition is satisfied, but there are still pathologies due to the violation of the weak energy condition. Within the classical context, scalar–tensor theories must be formulated in the Einstein conformal frame, not in the Jordan one.

Acknowledgments

This work was partially supported by EEC grants numbers PSS* 0992 and CT1*–CT94–0004, and by OLAM, Fondation pour la Recherche Fondamentale, Brussels.

References

- [1] Brans, C.H. and Dicke, R.H. (1961), *Physical Review* **124**, 925.
- [2] La, D. and Steinhardt, P.J. (1989), *Physical Review Letters* **62**, 376; Laycock, A.M. and Liddle, A.R. (1994), *Physical Review D* **49**, 1827.
- [3] Kolb, E.W., Salopek, D. and Turner, M.S. (1990), *Physical Review D* **42**, 3925; Steinhardt, P.J. and Accetta, F.S. (1990), *Physical Review Letters* **64**, 2740; Liddle, A.R. and Wands, D. (1992), *Physical Review D* **45**, 2665; Crittenden, R. and Steinhardt, P.J. (1992), *Physics Letters B* **293**, 32.
- [4] Weinberg, S. (1972), *Gravitation and Cosmology*, Wiley, New York.
- [5] Misner, C.W., Thorne, K.S. and Wheeler, J.A. (1973), *Gravitation*, Freeman, San Francisco.
- [6] Will, C.M. (1993), *Theory and Experiment in Gravitational Physics*, Cambridge University Press, Cambridge.
- [7] Magnano, G. and Sokolowski, L.M. (1994), *Physical Review D* **50**, 5039.
- [8] Bombelli, L. *et al.* (1987), *Nuclear Physics B* **289**, 735; Sokolowski, L.M. and Golda, Z.A. (1987), *Physics Letters B* **195**, 349; Sokolowski, L.M. (1989), *Classical and Quantum Gravity* **6**, 59; 2045; Cho, Y.M. (1990), *Physical Review D* **41**, 2462.
- [9] Cho, Y.M. (1992), *Physical Review Letters* **68**, 3133.
- [10] Cho, Y.M. (1994), in *Evolution of the Universe and its Observational Quest*, Yamada, Japan 1993, H. Sato ed., Universal Academy Press, Tokyo.
- [11] Cho, Y.M. (1997), *Classical and Quantum Gravity* **14**, 2963.
- [12] Taylor, T.R. and Veneziano, G. (1988), *Physics Letters B* **213**, 450; Cvetič, M. (1989), *Physics Letters B* **229**, 41; Ellis, J. *et al.* (1989), *Physics Letters B* **228**, 264; Damour, T. and Polyakov, A.M. (1994), *Nuclear Physics B* **423**, 532; (1994), *General Relativity and Gravitation* **26**, 1171.
- [13] Faraoni, V., Gunzig, E. and Nardone, P. (1998), *Fundamentals of Cosmic Physics* **20**, 121.

- [14] Wald, R.M. (1984), *General Relativity*, Chicago University Press, Chicago.
- [15] Witten, E. (1982), *Nuclear Physics B* **195**, 481.
- [16] Ford, L.H. and Roman, T.A. (1992), *Physical Review D* **46**, 1328; (1993), *ibidem* **48**, 776; (1995), *ibidem* **51**, 4277.
- [17] Faraoni, V. (1996), *Astrophysical Letters and Communications* **35**, 305.
- [18] Faraoni, V. and Gunzig, E. (1998), *Astronomy and Astrophysics* **332**, 1154.
- [19] Bracco, C. and Teyssandier, P., private communication.
- [20] Turner, M.S. (1993), in *Recent Directions in Particle Theory – From Superstrings and Black Holes to the Standard Model*, Proceedings of the Theoretical Advanced Study Institute in Elementary Particle Physics, Boulder, Colorado 1992, J. Harvey and J. Polchinski eds., World Scientific, Singapore.
- [21] Teyssandier, P. and Tourenç, P. (1983), *Journal of Mathematical Physics* **24**, 2793; Damour, T. and Esposito-Farèse, G. (1992), *Classical and Quantum Gravity* **9**, 2093; Damour, T. and Norvedt, K. (1993), *Physical Review Letters* **70**, 2217; (1993), *Physical Review D* **48**, 3436.
- [22] Scheel, M.A., Shapiro, S.L. and Teukolsky, S.A. (1995), *Physical Review D* **51**, 4236.